(7 pages) **Reg. No. :** ....

Code No.: 6366 **Sub. Code: ZMAM 13**

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2022.

First Semester

Mathematics – Core

ANALYTIC NUMBER THEORY

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If $n \mid n$, then it is called ________ property of divisibility.

   (a) reflexive

   (b) symmetric

   (c) transitivity

   (d) linearity

2. The series $\sum_{n=1}^{\infty} 1/p_n$ is ________

   (a) converges (b) diverges

   (c) countable (d) uncountable

3. The value $\varphi(8) =$ ________

   (a) 1 (b) 2

   (c) 4 (d) 0

4. The notation for Mobius function is ________.

   (a) $\varphi(n)$ (b) $\pi(n)$

   (c) $f(n)$ (d) $\mu(n)$

5. The identity function $I(n) = [1/n]$ is ________.

   (a) not multiplicative

   (b) multiplicative

   (c) completely multiplicative

   (d) not complete

6. If any two functions f and g are multiplicative, then ________ multiplicative

   (a) fg (b) f/g

   (c) none of the above (d) both

7. The value of $\lim_{x\to\infty} \frac{\pi(x)\log x}{x} =$ ________.

(a) 1 (b) 2

(c) 4 (d) 0

8. The average order of $\Lambda(n)$ is ________

(a) 2 (b) 1

(c) 4 (d) 0

9. The upper bound of $\pi(n) =$ ________.

(a) $\frac{1}{6}\frac{n}{\log n}$ (b) $\frac{n}{\log n}$

(c) $6\frac{n}{\log n}$ (d) $2\frac{n}{\log n}$

10. The value of $\lim_{x\to\infty} \frac{\log x}{\log \pi(x)} =$ ________.

(a) 0 (b) 2

(c) 4 (d) 1

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that every integer $n > 1$ is either a prime number or a product of prime numbers.

Or

(b) State and prove division algorithm.

12. (a) Prove that $\varphi(mn) = \varphi(m)\varphi(n)(d, \varphi(d))$ where $d = (m,n)$. Also prove that $\varphi(a) \mid \varphi(b)$ if $a \mid b$.

Or

(b) State and prove Mobius inversion formula.

13. (a) Given f with (1) = 1. Then prove that f is multiplicative if and only if $f\left(p_1^{a_1}, p_2^{a_2}, \dots, p_r^{a_r}\right) = f\left(p_1^{a_1}\right)f\left(p_2^{a_2}\right)\dots f\left(p_r^{a_r}\right)$ for all primes p_i and all integers $a_i \geq 1$.

Or

(b) State and prove Generalized inversion formula.

14. (a) If $x \geq 1$, then prove that $\sum_{n\leq x} \frac{1}{n} = \log x + C + 0\left(\frac{1}{x}\right)$. Also prove that $\sum_{n\leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + 0\left(x^{\alpha}\right)$ if $\alpha \geq 0$.

Or

(b) For all $x > 1$, show that $\sum_{n\leq x} \varphi(n) = \frac{3}{\pi^2}x^2 + 0(x\log x)$.

15. (a) For all $x \geq 1$, prove that $\left|\sum_{n \geq x} \frac{\mu(n)}{n}\right| \leq 1$ with equality holding only if $x < 2$.

Or

(b) For $x \geq 2$, show that $\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + 0(x)$ where the sum is extended over all primes $\leq x$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)

16. (a) (i) State and prove Euclidean algorithm.

(ii) If $(a,b) = 1$, then prove that $(a^n, b^k) = 1$ for all $n \geq 1$, $k \geq 1$.

Or

(b) (i) Prove that $n^4 + 4$ is composite if $n > 1$.

(ii) Prove that every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.

17. (a) State and prove the product formula for $\varphi(n)$.

Or

(b) Define Mobius function and find the relationship between Mobius function and Euler totient function.

18. (a) Define Liouville's function and for every $n \geq 1$ and prove that $\sum_{d|n} \lambda(d) = \begin{cases} 1, & \textit{if } n \textit{ is a square} \\ 0, & \textit{otherwise} \end{cases}$.

Or

(b) State and prove Generalized Mobius inversion formula.

19. (a) Prove that the set of lattice points visible from the origin has density $6/\pi^2$.

Or

(b) For all $x \geq 1$ and $\alpha > 0$, $\alpha \neq 1$, prove that,

(i) $\sum_{n \leq x} \sigma_1(n) = \frac{1}{2}\zeta(2)x^2 + 0(x \log x)$

(ii) $\sum_{n \leq x} \sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + 0(x^\beta)$ where $\beta = \max\{1, \alpha\}$.

20. (a) For $n \geq 1$, prove that the n^{th} prime p_n satisfies the inequality $\frac{1}{6} n \log n < p_n < 12\left(n \log n + n \log \frac{12}{e}\right)$.

Or

(b) Prove that the following relations are logically equivalent

(i) $\lim_{x\to\infty} \dfrac{\pi(x)\log x}{x} = 1$

(ii) $\lim_{x\to\infty} \dfrac{\vartheta(x)}{x} = 1$

(iii) $\lim_{x\to\infty} \dfrac{\psi(x)}{x} = 1$